NONLINEAR W_∞ ALGEBRAS FROM NONLINEAR INTEGRABLE DEFORMATIONS OF SELF DUAL GRAVITY

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Abstract

A proposal for constructing a universal nonlinear $\hat{W}_\infty$ algebra is made as the symmetry algebra of a rotational Killing-symmetry reduction of the nonlinear perturbations of Moyal-Integrable deformations of $D=4$ Self Dual Gravity (IDSDG). This is attained upon the construction of a nonlinear bracket based on nonlinear gauge theories associated with infinite dimensional Lie algebras. A Quantization and supersymmetrization program can also be carried out. The relevance to the Kadomtsev-Petviashvili hierarchy, $2D$ dilaton gravity, quantum gravity and black hole physics is discussed in the concluding remarks.

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1 Introduction

Recently [1] a universal linear W_∞ was constructed as the symmetry algebra of a rotational Killing-symmetry reduction of the Moyal integrable deformations of $D=4$ Self Dual Gravity (IDSDG) [2]. This algebra turned out to be the *complexification* of the direct sum of the chiral and antichiral $W_\infty, \bar{W}_\infty$ algebras. Central extensions can be constructed by the cocycle formula in terms of the logarithm of the derivative operator discussed in [3]. The latter work was a complement of [4] where Fairlie and Nuyts using Moyal brackets found a basis of differential operators on the circle which yield the structure constants of the centerless linear W_∞ [5]. An epimorphism map (onto but not $1-1$ map) was established from the universal algebra into the nonlinear $\hat{W}_\infty$ algebra of Wu and Yu [6] although the explicit solution was

not presented in [1]. A one parameter family of Hamiltonian strutures for the Kadomtsev-Petviashvili (KP) hierarchy and a continous deformation of nonlinear W_{KP} algebras has been given by [7]. What is required now is a unifying geometrical framework to generate these algebras.

The purpose of this letter is to construct nonlinear perturbations of the $IDSDG$ based on a nonlinear bracket providing solutions to the epimorphisms of [1]. Such nonlinear bracket is based on a nonlinear gauge theory principle [8] which has attracted attention recently in connection to $2D$ dilaton supergravity theories and Yang-Mills-like formulation of R^2 gravity with dynamical torsion generalizing the Jackiw-Teitelboim's model. The nonlinear algebra (strictly speaking these are not Lie algebras) is based on the complexification of the area-preserving diffs of a two-dim surface, $sdiff\ \Sigma$, which is essentially $SU^*(\infty) \sim SL(\infty, H)$ [9,10,20]. (quaternionic valued). Self Dual Yang-Mills equations (SDYM) based on this group, after a suitable dimensional reduction and ansatz, lead to Plebanski's second heavenly equation which furnishes solutions of SDG in $D = 4, 2+2$. A Darboux change of variables converts the latter equation into the first heavenly equation whose rotational Killing-symmetry reduction is linked to W_∞ algebras via the $SL(\infty)$ continous Toda equation [11]. Therefore, a nonlinear gauge theory based on $SU^*(\infty)$ SDYM in four dimensions is linked to nonlinear $\hat{W}_\infty$ algebras [6]. It is in this fashion how we construct the new bracket which incorporates the nonlinearities of [6].

The importance of Infinite Dimensional Lie Algebras and the Geometry of Integrable Systems in connection to Self Dual Gravity has been emphasized by many authors, in particular by Park [11], Strachan and Takasaki [2,12]. The supersymmetric extension was carried out by the author [13]. Since the $sdiff\ \Sigma$ Lie algebra preserve the Poisson symplectic structure (the area in phase space) an integrable deformation of these models was possible by means of the Moyal bracket [14] while, at the same time, retaining the geometrical ideas central to the integrability of SDYM/SDG. The connection with the theory of integrable systems came from the observation by Atiyah [15] that these are reductions of SDYM systems in four or higher dimensions. For further mathematical details pertaining the geometry of SDYM see Ward [16].

A non-linear extension of these algebras and their quantization was provided by Wu and Yu [6] by quantizing the conformal non-compact coset model $sl(2,R)_k/U(1)$ in which the $\hat{W}_\infty$ algebra appears as a hidden current algebra (Bakas and Kiritsis [18]). Because this $sl(2,R)_k/U(1)$ coset model is connected to Witten's black hole in $2D$ string theory, its quantization plays an important role in understanding the physics of two-dim quantum gravity and black-holes [19] .It is for this reason that a geometrical construction of a Universal nonlinear $\hat{W}_\infty$ algebra is warranted. This is the purpose of this work.

In II [1] a candidate for a Universal linear W_∞ algebra is obtained from a Killing-symmetry reduction of Moyal-integrable deformations of Self Dual Gravity (IDSDG) in $D = 4$ based on the ideas by Strachan and Takasaki [2,12]. In III we present the nonlinear bracket extension of the Moyal algebra and postulate the form of the equations governing the nonlinear perturbations of Moyal integrable deformations of SDG and the algebra of symmetry transformations. Finally, some brief remarks are made in relation to black-hole physics and canonical quantum gravity. Complexified $D = 4$ spacetime is always assumed. C^4 is parametrized by the variables : $y, z, \tilde{y}, \tilde{z}$. Moyal brackets are always assumed unless otherwise indicated. Plebanski's first heavenly equation is $\{\Omega_{,y}\, , \Omega_{,z}\,\}_{\tilde{y},\tilde{z}} = 1$; where the

bracket is the Poisson one. Solutions of this equation yield complexified self dual metrics of the form : $ds^2 = \Omega_{i\tilde{j}} dx^i d\tilde{x}^j$. $x^i = y, z;$ *and* $\tilde{x}^j = \tilde{y}, \tilde{z}$.

2 The Universal W_∞ Algebra

We begin with by writing down the derivatives with respect to $y, \tilde{y}$ (when acting on Ω) which appear in the Moyal integrable deformation of Plebanski's first heavenly equation to be discuss below. $r \equiv y\tilde{y}$.

$$\partial_y = (1/y) r\partial_r.\ \partial_{\tilde{y}} = (1/\tilde{y}) r\partial_r.\ \partial_y \partial_{\tilde{y}} = r\partial_r^2 + \partial_r. \tag{1}$$

$$(\partial_{\tilde{y}})^2 = (1/\tilde{y})^2 (r^2\partial_r^2 + r\partial_r).\ \ (\partial_{\tilde{y}})^3 = (1/\tilde{y})^3 (r^3\partial_r^3 + r^2\partial_r^2 - r\partial_r)...... \tag{2}$$

The Moyal bracket is obtained as a power expansion in the deformation parameter κ of the expression :

$$\{f, g\}_{Moyal} \equiv [\kappa^{-1} sin\ \kappa(\partial_{\tilde{y}_f}\partial_{\tilde{z}_g} - \partial_{\tilde{y}_g}\partial_{\tilde{z}_f})] fg. \tag{3}$$

where the subscripts under $\tilde{y}, \tilde{z}$ imply that the derivatives act only on f or on g accordingly. In view of these higher order derivatives in the Moyal bracket one can see that one is going to have a compatible power expansion (of the solutions to the deformations of Plebanski's first heavenly equation [11,12]) with the rotational Killing symmetry condition : $\Omega(r \equiv y\tilde{y}; z, \tilde{z})$ if, and only if, one has the following power expansion for Ω :

$$\Omega(y, \tilde{y}, z, \tilde{z}) \equiv \sum_{n=0}^{\infty} (\kappa/\tilde{y})^n \Omega_n(r, z, \tilde{z}) \tag{4}$$

where each Ω_n is a function *only* of $r, z, \tilde{z}$. Plugging this expansion into the integrable deformation of Plebanski's first heavenly equation :

$$\{\Omega_z, \Omega_y\}_{Moyal} = 1. \tag{5}$$

where the Moyal bracket is taken with respect to $\tilde{z}, \tilde{y}$ yields an infinite family of differential equations with respect to the variables $r, z, \tilde{z}$ only :

$$\begin{gathered}\{\Omega_{0z}, \Omega_{0y}\}_{Poisson} = 1. \\ 0 = \Omega_{0z\tilde{z}}[-\Omega_{1r} + (r\Omega_{1r})_r] - r\Omega_{0rz}\Omega_{1r\tilde{z}} + \\ \Omega_{1z\tilde{z}}(r\Omega_{0r})_r + \Omega_{0r\tilde{z}}(\Omega_{1z} - r\Omega_{1rz}). \\ ....\end{gathered} \tag{6}$$

where the subscripts represent partial derivatives of the functions $\Omega_0, \Omega_1, ...$ with respect to the variables $r, z, \tilde{z}$ in accordance with the Killing symmetry reduction condition. The first equation, after a change of variables, *is* nothing but the $sl(\infty)$ continual Toda equation [11] whose symmetry algebra is the linear classical w_∞ algebra [3]. The rest of the equations are then the Moyal integrable deformations of the continual Toda equation.

In order to determine the symmetry algebra of this infinite family we follow closely Strachan and Takasaki who showed that an infinitesimal symmetry of equation (5) must be

a solution of the deformation of Laplace's equation with respect to a background-solution of (5) (upon apllication of δ on (5)) :

$$\Delta_\Omega \delta\Omega \equiv \{\Omega_z, \delta\Omega_y\} - \{\Omega_y, \delta\Omega_z\} = 0. \tag{7}$$

where Ω is a solution of (5).

A general straightforward solution of (7) which was not given by [2,12], since these authors did not discuss Killing symmetry reductions of (5), is :

$$\delta\Omega \equiv \alpha[\epsilon L_\Lambda, \Omega] = \{\Lambda(\tilde{y}, \tilde{z}), \Omega\} = \sum_{n=0} (\kappa/\tilde{y})^n \delta\Omega_n(r, z, \tilde{z}). \tag{8}$$

due to the antisymmetry and Jacobi properties of the Moyal bracket and to the fact that Ω is a solution of (5). α is a parameter whose dimension is $(length)^2$ and it is needed to match dimensions. It will be set to *unity* in all of the equations that follow. Λ is taken to be dimensionless. ϵ is an infinitesimal parameter corresponding to the infinitesimal symmetry and L_Λ is a generator of the universal W_∞ algebra to be determined below. The generators of the universal algebra are "parametrised" by a family of functions :

$$\Lambda(\tilde{y}, \tilde{z}) = \sum_{n=0} (\kappa/\tilde{y})^n \tilde{y} f_n(\tilde{z}). \tag{9}$$

After performing the Moyal bracket in (8) one has an infinite family of equations yielding the transformations $\delta\Omega_n(r, z, \tilde{z}), n = 0, 1, 2....$:

$$\begin{aligned} \delta\Omega_0 &= r\Omega_{0r} f_{0\tilde{z}} - f_0 \Omega_{0\tilde{z}}. \\ \delta\Omega_1 &= -f_0 \Omega_{1\tilde{z}} - f_{0\tilde{z}} \Omega_1 + r\Omega_{1r} f_{0\tilde{z}} + r\Omega_{0r} f_{1\tilde{z}}. \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{10}$$

The universal symmetry algebra is furnished as the Lie algebra of derivations on the space of solutions of (5):

$$[\delta_{\Lambda^1}, \delta_{\Lambda^2}]\Omega = \delta_{\Lambda^3}\Omega = \delta_{\Lambda^1 \otimes \Lambda^2}\ \Omega. \tag{11}$$

where $\Lambda^1 \otimes \Lambda^2 \equiv \{\Lambda^1, \Lambda^2\}$. The Lie algebra of derivations is consistent with the Lie-Moyal bracket structure present in the physics of Moyal integrable deformations of Self Dual Gravity :

$$L_{\Lambda^1} \otimes L_{\Lambda^2} = [L_{\Lambda^1}, L_{\Lambda^2}] = L_{\Lambda^1 \otimes \Lambda^2} = L_{\{\Lambda^1, \Lambda^2\}}. \tag{12}$$

Given two functions $\Lambda^1(\tilde{y}, \tilde{z}), \Lambda^2(\tilde{y}, \tilde{z})$ the Lie-Moyal structure yields Λ^3 in terms of the former two functions after expanding in powers of $(\kappa/\tilde{y})^n$:

$$\begin{aligned} f_0^3 &= f_0^2 f_{0\tilde{z}}^1 - f_0^1 f_{0\tilde{z}}^2 \neq 0. \\ f_1^3 &= f_0^2 f_{1\tilde{z}}^1 - f_0^1 f_{1\tilde{z}}^2 \neq 0. \\ f_2^3 &= (f_2^1 f_{0\tilde{z}}^2 - 1 \leftrightarrow 2) + (f_0^2 f_{2\tilde{z}}^1 - 1 \leftrightarrow 2) \neq 0. \\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{13}$$

Where we should keep in mind always the presence of the α parameter which was set to *one* in the above equations so that the dimensions match properly. Therefore, iteratively, we solve for the rotational Killing-symmetry reduction of the Moyal-integrable deformations of Self Dual Gravity (IDSDG) in $D = 4$ and find its infinite dimensional universal W_∞ algebra which is the Moyal deformation of the w_∞ algebra associated with the $sl(\infty)$ continual Toda equation. Park [11] proved that this algebra is in fact larger and was the $w_\infty \oplus \bar{w}_\infty$ (after a suitable real slice). Hence, the candidate universal linear W_∞ algebra is the *complexification* of $W_\infty \oplus \bar{W}_\infty$. Clearly this algebra is much bigger than W_∞.

3 The Nonlinear W_∞ Algebra

We shall define the new bracket which incorporates the nonlinearities of the W algebras symbolically as follows. Given two gauge potentials their bracket is :

$$[A_\mu, A_\nu] \equiv \frac{\{A_\mu, A_\alpha\}}{1 - \lambda F^{-1}_{\alpha\beta}\{A_\beta, A_\nu\}}. \tag{14}$$

A power expansion in λ yields :

$$[A_\mu, A_\nu] \equiv \{A_\mu, A_\nu\} + \lambda\{A_\mu, A_\alpha\}F^{-1}_{\alpha\beta}\{A_\beta, A_\nu\} + \lambda^2\{A_\mu, \}F^{-1}\{,\}F^{-1}\{, A_\nu\} + ... \quad (15).$$

$F^{-1}_{\alpha\beta}$ is the *inverse* matrix function associated with the field-strength of the $SU^*(\infty)$ Yang-Mills theory. We recall that $SU^*(\infty)$ Yang-Mills theory [20]; i.e.the gauge theory of the $sdiff\ \Sigma$ group requires replacing Lie-algebra valued gauge potentials by c-number $A_\mu(x^m; q, p)$ objects depending on the two extra internal variables parametrising the surface Σ in addition to x^m. Lie-algebra commutators are replaced by Poisson brackets.

The bracket (15) is reminiscent of Dirac's bracket in the quantization of gauge systems with constraints. The bracket *linearizes* in the special case that the gauge potentials are spacetime *independent* :

$$F_{\mu\nu} \equiv \partial_\nu A_\mu - (\mu \leftrightarrow \nu) + \{A_\mu, A_\nu\} \Rightarrow \{A_\mu, A_\nu\}. \tag{16}$$

The bracket is the Moyal bracket with respect to the internal variables if we wish to make contact with the nonlinear $\hat{W}_\infty$ algebra of Wu and Yu [6]. Summation over repeated indices is implied. The bracket's antisymmetry and derivation properties are trivially satisfied by inspection.$[A_\mu, A_\nu] = -[A_\nu, A_\mu]$ and $[A, BC] = [A, B]C + B[A, C]$. This is due to the bracket nature of the last Moyal bracket in each one of the terms in the r.h.s of (15). The Jacobi property is more subtle and can be verified if one recurs to the vector calculus result :

$$\vec{A} \wedge (\vec{B} \wedge \vec{C}) + cyclic = 0. \tag{17}$$

Per example, the term :$\{A, M\}F^{-1}_{MN}\{N, B\}$ ca be represented symbolically, after a matrix multiplication of three antisymmetric matrices, as :

$$(\vec{A} \wedge \vec{B})\phi(\vec{M}, \vec{N}). \tag{18}$$

ϕ is a scalar valued function of $\vec{M}, \vec{N}$. One learns that $\phi(\vec{M}, \vec{N}) = \phi(\vec{N}, \vec{M})$. This is due to the fact that the bracket $[,]$ is a map from $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. The vector space of the gauge

potentials is represented by $\mathcal{A}$. This explains why one must contract vector indices as shown in eq-(18). It is for this reason that one must have insertions of the *inverse* matrix function between two Moyal brackets exactly as it occurs in Dirac's prescription for quantization of constrained Hamiltonian systems. A structure similar to (14,15) appeared in the free field realizations of $\hat{W}_\infty$ given in [6,18] where an explicit form of the KP first order pseudo-differential operator was given as ($D = d/dz$) :

$$L = D + \sum_{r=0}^{\infty} u_r D^{-r-1} = D + \bar{j}\frac{1}{D-(\bar{j}+j)}j. \tag{19}$$

Therefore the bracket $[A,[B,C]]$ can be represented symbolically as a sum of terms of the form :

$$\vec{A} \wedge (\vec{B} \wedge \vec{C})[1+\phi+\phi^2+....]^2 \tag{20}$$

which upon taking the cyclic sum one gets zero. Since the $\hat{W}_\infty$ algebras are arbitrarily nonlinear (in powers of λ) one must sum over an arbitrary number of terms. Setting $\lambda = 0$ one recovers the Moyal algebra which was shown to be isomorphic to the centerless linear W_∞ algebra in [3,4]. Because the symmetry algebra is the *complexification* of the direct sum of a chiral and antichiral $W_\infty, \bar{W}_\infty$ algebra, respectively [1], in order to establish the isomorphism, one first has to take a suitable real slice and then project into the chiral sector. The symmetry algebra of the nonlinear perturbations of the Moyal integrable deformations of Self Dual Gravity (after a Killing symmetry reduction) is larger than the $\hat{W}_\infty$ algebra. For this reason we coined them "universal" W_∞ algebras [1]. Central terms can be incorporated by adding cocycles to the algebras and these can be expressed in terms of the logarithm of the derivative operator given in [3]. Cocycles in terms of the symmetries of the *tau* function were also given previously by Takasaki [12]. If one had used Poisson brackets, instead, one would obtain nonlinear deformations of the Bakas algebra w_∞. A Quantization program can be carried out by performing a quantum deformation of the algebra as it was performed in [6] to obtain the Quantum $\hat{W}_\infty$ algebra.

Having written down the new bracket we can conjecture that the nonlinear perturbations of Moyal integrable deformations of Self Dual Gravity can be written in terms of the Plebanski 's second heavenly form $\Theta(y,\tilde{y},z,\tilde{z})$ as follows [9,10]. Start with the complexified $SU^*(\infty)$ Self Dual Yang-Mills equations in complexified spacetime, C^4, of signature $(4,0),(2,2)$, respectively , $F_{y\tilde{y}} + (-)F_{z\tilde{z}} = 0$ and $F_{yz} = F_{\tilde{y}\tilde{z}} = 0$. The curvatures must be computed using the nonlinear bracket (14,15). We learnt from [9,10] that, in the case of ordinary Poisson brackets, a dimensional reduction : $\partial_y = \partial_{\hat{q}}$; $-\partial_{\tilde{y}} = \partial_{\hat{p}}$ and the ansatz of [9,10], where the hatted variables $\hat{q},\hat{p}$ represent the complexification of the $sdiff\ \Sigma$, yields Plebanski's second heavenly equation [9,10] :

$$(\Theta,_{\hat{p}\hat{q}})^2 - \Theta,_{\hat{p}\hat{p}}\Theta,_{\hat{q}\hat{q}} + \Theta,_{z\hat{q}} - \Theta,_{\tilde{z}\hat{p}} = 0. \tag{21}$$

The semicolon means partial derivatives with respect to the corresponding variables. Eq-(21) yields self-dual solutions to the complexified Einstein's equations providing a family of hyper-Kahler metrics on the complexification of the cotangent space of Σ. Θ represents a continuous self dual deformation of the flat metric in $T^*\Sigma$. The field-strengths are now computed with the nonlinear bracket given by (15) so that (21) is modified accordingly. The

symmetry algebra of the space of solutions of (21) originates from the original YM gauge invariance of the $SU(\infty)$ SDYM equations :

$$\delta A_\mu = D_{A_\mu}\epsilon.\ \delta F_{\mu\nu} = [F_{\mu\nu}, \epsilon]. \tag{22}$$

The YM potentials, as functions of Θ, were given in [9,10] : $A_y = \Theta,_y$. $A_{\tilde{y}} = \Theta,_{\tilde{y}}$ ..... Therefore, symmetries under the transformations (22) determine those of Θ. Due to the nonlinearities of (15) the field strength does not transform homogeneously unless the gauge parameter is a suitable judicious field dependent quantity, $\epsilon(A)$ obeying an equation like the one (26) below.

Where do the W symmetries appear ? From the underlying KP equation . A detailed construction was given in [9] mapping solutions of the KP equation into the dimensionally-reduced solutions (from 4 to 3) of the $D = 2 + 2$ Plebanski's second heavenly equation after a certain tuning (constraints) between the YM potentials was imposed. The dimensionally-reduced Plebanski equation turned out to be after one showed that $F_{y\tilde{y}} = 0$:

$$F_{z\bar{z}} = \Theta,_{zq} - \Theta,_{\bar{z}p} = 0. \tag{23}$$

We will see shortly the importance of (23). When (15) is implemented one has a deformation of (23) :$\mathcal{F}_{z\bar{z}} = 0$. The latter implies that (23) receives λ corrections so that $F_{z\bar{z}}$ is no longer zero. A zero value for F will also render eq-(15) ill-defined : it will be singular.

Boer and Goeree in a series of articles [21] constructed a covariant W gravity action and provided with a geometrical interpretation of W transformations (a W algebra) from Gauge Theory for those W algebras related to embeddings of $sl(2)$ in a Lie algebra $\mathbf{g}$. The W transformations were just homotopy contractions of ordinary gauge transformations. Their starting point was to constrain the A_z potential to have the form :$A_z = \mathbf{\Lambda} + \mathbf{W}$ where Λ is a constant matrix and the components of $\mathbf{W}$ transform as fields of certain spins determined by a gradation of $\mathbf{g}$:

$$\mathbf{W} = \sum W^i(z,\bar{z})X_i.\ \mathbf{g} = \oplus g_{\alpha_i}. \tag{24}$$

X_i are the generators within each member g_{α_i} of the gradation and W^i are the generators of the W algebra. So we can see how the YM potentials contain the W generators (currents).

The second step was to find those gauge transformations that leave the form of A_z invariant :

$$\delta A_z = D_{A_z}(A_{\bar{z}}).\ A_{\bar{z}} = \frac{1}{1 + \lambda L(\partial + Ad_W)} Kernel(Ad_\Lambda). \tag{25}$$

λ is a parameter which will be identified with the one in (15). L is the homotopy ("integration") operator which roughly maps Lie-algebra-valued p forms in R^n to $(p-1)$ forms on R^{n-1}. It is defined as the inverse of the adjoint action of Λ : $Ad_\Lambda = [\Lambda, *]$. The expression relating $A_{\bar{z}}$ in terms of the sum of a series of operators acting on the kernel of the Ad_Λ; i.e. in terms of A_z, was obtained from the zero curvature condition :$F(A_z, A_{\bar{z}}) = 0$. Now we can see the relevance of eq-(23). Finally, what is required is to find the form of the $field-dependent$ gauge parameter :$\epsilon(A)$ appearing in (22) in order to match the W transformations of (25) in the $N \rightarrow \infty$ limit. We shall refer now to $sl(\infty, H) \sim su^*(\infty)$. One has :$\mathbf{\Lambda} \rightarrow \Lambda(q,p).\mathbf{W} \rightarrow W(z,\bar{z},q,p)$....

Writing the expression for $A_{\bar{z}}$ as $A^o_{\bar{z}} + \sum \lambda^n A^{(n)}_{\bar{z}}$; and a similar expression for $\epsilon(A) = \sum \lambda^n \epsilon^{(n)}(A)$ allows to determine, order by order in λ, the required form of each of the terms appearing in the expansion of $\epsilon(A)$:

$$\partial_z \epsilon^o + \{A_z, \epsilon^o\} = (\partial_z + \{A_z, *\})A^o_{\bar{z}}.$$

$$\partial_z \epsilon^1 + \{A_z, \epsilon^1\} + \{A_z, A_\alpha\}F^{-1}_{\alpha\beta}\{A_\beta, \epsilon^o\} = (\partial_z + \{A_z, *\})A^1_{\bar{z}}.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \tag{26}$$

The YM potentials appearing in the l.h.s of (26) are solutions to :$\mathcal{F}_{z\bar{z}} = 0$ and should not be confused with the solutions to the $F_{z\bar{z}} = 0$. Eq-(26) will ensure that $\mathcal{F}$ transforms as it should since (26) is part of

$$\delta_W F_{z\bar{z}} = \delta \mathcal{F}_{z\bar{z}} = [\mathcal{F}_{z\bar{z}}, \epsilon(A)] = 0. \tag{26b}$$

due to the fact that $\mathcal{F}_{z\bar{z}} = 0$. For consistency one ought to add a holomorphic function $f(z, \lambda)$ to the r.h.s of (26). The brackets appearing in (26) are Poisson brackets. Upon quantization one replaces them by the Moyal brackets. This we learnt from [5] that a quantization of w_∞ deforms into the linear W_∞. The brackets are computed from the relationship :

$$\{f(A_i), g(A_j)\} = \frac{\delta f}{\delta A_i}\{A_i, A_j\}\frac{\delta g}{\delta A_j}. \tag{27}$$

Given two solutions ϵ_1, ϵ_2 to the functional differential equation given by (26) one has, by construction, that these W transformations satisfy a nonlinear W_∞ algebra.

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\mathcal{F} = \delta_{\epsilon_3}\mathcal{F} = 0.\ \epsilon_3 = [\epsilon_1, \epsilon_2]. \tag{28}$$

This is due to the Jacobi property of the bracket (15). If one wishes to recast the above in terms of Plebanski's first heavenly equation one can. A Darboux change of variables transforms the former equation (21) into Plebanski's first heavenly equation. When a Moyal bracket is used we learnt [2] that the latter equation deforms into $\{\Omega_z, \Omega_y\} = 1$. When the nonlinear bracket is being used, instead, such equation should represent the nonlinear perturbations associated with the Moyal deformations of Self Dual Gravity. This equation reads

$$[\Omega_z(A_\mu), \Omega_y(A_\nu)] = 1. \tag{29}$$

In order to compute this last bracket in terms of (15) one must insert the dependence of Ω on the potentials due to the Darboux change of coordintes which is essentially a Backlund-type of transformation relating derivatives of Θ to those of Ω. Since the YM potentials are essentially derivatives of Θ the connection may be established in order to evaluate (28). One must also include integration "constants" (functions) which are not purely arbitrary because one has to satisfy (28) as well as the nonlinear generalization of (21) which constrains the possible solutions of the YM potentials. A further rotational Killing symmetry reduction of (28) in the lines of II yields an infinite family of equations whose symmetry algebra is the *complexification* of $\hat{W}_\infty \oplus \overline{\hat{W}}_\infty$. This is the main result of this letter completing the results in [1]. To conclude : Nonlinear Gauge Theories [8] of the $SU^*(\infty)$ SDYM theory

contain (after dimensional reductions and/or Killing symmetry reductions) the nonlinear $\hat{W}_\infty$ algebras. A quantization of the Killing symmetry reductions of nonlinear perturbations of the $IDSDG$ in $D = 4$ can be achieved thanks to the Quantum $\hat{W}_\infty$ algebra constructed recently by Wu and Yu [6]. This quantum algebra contains all algebras in the classical limit after apropriate truncations and contractions. It was seen as a realization of the hidden symmetries of the quantized non-compact coset model $sl(2,R)_k/U(1)$ [18].

Perhaps the most important consequence of the quantization based on the $sl(2,R)_k/U(1)$ coset model is the connection of this coset model to the black hole in $2D$ string theory found by Witten [19]. The construction of the Quantum version of the KP hierachy provided for us an infinite set of commuting quantum charges in an explicit and closed form. It is not difficult to see the importance that these infinite number of conserved Quantum charges will have for the Quantum $\hat{W}_\infty$ $Hairs$ of Witten's $2D$ stringy-black hole solution associated with the gauged-$sl(2,R)_k/U(1)$ WZNW models. What would be the situation in the full-fledged $4D$ Gravitational theory versus the Killing-symmetry reductions of nonlinearly-perturbed $IDSDG$ (an effective $3D$ theory)? The work of Ashtekar and others [22] on the loop representation of canonical quantum gravity overlaps with ours in the sense that we had started with a complexified self dual $SU^*(\infty)$ Yang-Mills theory which led to Self Dual complexified Gravity upon dimensional reduction [9,10]; and the effective $3D$ theory upon the Killing-symmetry reduction should bear a connection to the Knot/Chern-Simmons theoretical formulations of $3D$ gravitational theory. The fact that it is $SU^*(\infty)$ SDYM theory the one that generates the lower dimensional integrable models and their hierarchies points that its moduli space must contain important information pertaing the Geometry of Strings, IDSDG, black holes, etc....To conclude : IDSDG +nonlinear-integrable perturbations+Quantization $\sim$ Quantum-Nonlinear W_∞ algebra.

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